Rare-Earth Nuclei: Radii, Isotope-Shifts and Deformation Properties in the Relativistic Mean Field Theory

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Abstract

A systematic study of the ground-state properties of even-even rare earth nuclei has been performed in the framework of the Relativistic Mean-Field (RMF) theory using the parameter set NL-SH. Nuclear radii, isotope shifts and deformation properties of the heavier rare-earth nuclei have been obtained, which encompass atomic numbers ranging from Z=60 to Z=70 and include a large range of isospin. It is shown that RMF theory is able to provide a good and comprehensive description of the empirical binding energies of the isotopic chains. At the same time the quadrupole deformations β_2 obtained in the RMF theory are found to be in good agreement with the available empirical values. The theory predicts a shape transition from prolate to oblate for nuclei at neutron number N=78 in all the chains. A further addition of neutrons up to the magic number 82 brings about the spherical shape. For nuclei above N=82, the RMF theory predicts the well-known onset of prolate deformation at about N=88, which saturates at about N=102. The deformation properties display an identical behaviour for all the nuclear chains. A good description of the above deformation transitions in the RMF theory in all the isotopic chains leads to a successful reproduction of the anomalous behaviour of the empirical isotopic shifts of the rare-earth nuclei. The RMF theory exhibits a remarkable success in providing a unified and microscopic description of various empirical data.

1 INTRODUCTION

The Relativistic Mean-Field (RMF) theory [1, 2] has proved to be a powerful tool for an effective microscopic description of nuclear structure at and away from the line of stability. With a very limited number of parameters in a non-linear [3] nuclear Lagrangian, one is able to obtain a very good quantitative description of the ground-state properties of spherical and deformed nuclei [4, 5, 6] at and away from the stability line [7, 8, 9]. Successful attempts have also been made to obtain dynamical properties such as collective excitations [10] and identical bands in rotating superdeformed nuclei [11].

In the RMF theory the saturation and the density dependence of the nuclear interaction is obtained by a balance between a large attractive scalar σ-meson field and a large repulsive vector ω-meson field. The asymmetry component is provided by the isovector ρ meson. The nuclear interaction is hence generated by the exchange of various mesons between nucleons in the framework of the mean field. The spin-orbit interaction arises naturally in the RMF theory as a result of the Dirac structure of nucleons.

In an earlier work [7] we studied the isotope shifts of Pb nuclei. It was shown that the anomalous kink at the shell closure (at N=126) [7] could be described within the RMF theory successfully. This long standing problem remained a puzzle within the non-relativistic mean-field approach using various Skyrme type of forces [12]. The spin-orbit term in the RMF theory, which is different from that in the Skyrme approach, lies at the origin of this success. Inspired by the success of the RMF theory to explaining various subtle aspects originating from shell effects, the form of the spin-orbit potential in the Skyrme theory has been modified and a Modified Skyrme Ansatz (MSkA) has been proposed [13, 14],

which includes only the one-body (direct) component of the spin-orbit force in the Skyrme theory vis-a'-vis the usual Skyrme theory which includes also the exchange term in the spin-orbit potential. Consequently, this has led to a success in obtaining the kink in the isotope shifts of Pb nuclei in the MSkA. The broader implications of the MSkA are under investigation. Another approach to tackling this problem of the kink in the isotope shifts was undertaken in the framework of conventional Skyrme theory in Ref. [15] by introducing an additional parameter in the two-body spin-orbit force.

Recently a systematic study [6] of ground-state properties of Kr, Sr and Zr isotopes was performed in the framework of the RMF theory using the force NL-SH [7]. It was shown that the RMF theory provides a good description of the binding energies, charge radii and deformation properties of nuclei in the Z=40 region over a large range of isospin. The RMF theory has predicted a complex web of many dramatic shape transitions in the isotopic chains of Sr, Kr and Zr. A shape coexistence in several heavy Sr isotopes has also been seen. It was observed that the RMF theory is able to describe the anomalous kinks in isotope shifts in Kr and Sr nuclei, the problem which was hitherto unsolved [16, 17]. In this case deformation changes play an important role to understand the experimental isotope shifts, and a combined effect of the theory to be able to reproduce deformation and the inherent shell effects in the RMF theory lead to the above results.

The rare-earth nuclei have been of great experimental as well as theoretical interest. Most of the studies are confined to employing the approach of Nilsson-Strutinsky [18] or using the Skyrme Ansatz for deformed nuclei [19]. Self-consistent calculations with Skyrme or Gogny forces were performed for a number of rare-earth isotopes [19, 20, 21, 22, 23, 24, 25, 26]. Relativistic calculations with

the force NL1 [27] have also been performed for a few representative rare-earth nuclei [4]. However, in most of these studies only some specific rare-earth nuclei were investigated, hindered largely by large computing time needed for such calculations and also due to uncertainties in the available interactions. In this investigation we use the parameter set NL-SH, which provided a very good description of the ground-state properties of deformed Xe nuclei [5] and of nuclei in the region of Z=40 [6]. We study systematically, the ground-state properties of six isotopic chains of rare-earth nuclei with atomic numbers $60 \leq Z \leq 70$ covering a large range of masses. This is aimed at drawing some general conclusions about the properties of these nuclei as well as the trend of their variation with the neutron number. We focus our attention mainly on the sizes, the isotope shifts and the quadrupole deformations due to the special interest they present. It is worth noting that the isotope shifts and the deformations of the heavier rare-earth nuclei have been measured systematically by Neugart et al. [28, 29, 30, 31] and also by the Lenigrad-Troitz collaboration [32, 33, 34]. In addition experimental data are available on the quadrupole and hexadecapole moments of various rare-earth nuclei. (see for example [35, 36] and also [37, 38, 39, 40, 41, 42, 43, 44]).

In this work we present a detailed and exhaustive microscopic study of the ground-state properties of several isotopic chains of nuclei in the rare-earth region. In section II we give some details of the formalism of the RMF theory used for deformed nuclei. In section III we provide some details of the calculations. In section IV we present and discuss our results. A comparison of the RMF results is made with empirical data wherever available. The last section summarizes our main conclusions.

2 THE RELATIVISTIC MEAN-FIELD FORMALISM

The starting point of the RMF theory is a Lagrangian density [1, 2] which describes the nucleons as Dirac spinors interacting via the exchange of several mesons. The Lagrangian density can be written in the following form:

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(i\not{\partial} - M)\psi + \tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - U(\sigma) - \tfrac{1}{4}\Omega_{\mu\nu}\Omega^{\mu\nu} + \\ &\tfrac{1}{2}m_\omega^2\omega_\mu\omega^\mu - \tfrac{1}{4}\vec{R}_{\mu\nu}\vec{R}^{\mu\nu} + \tfrac{1}{2}m_\rho^2\vec{\rho}_\mu\vec{\rho}^\mu - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &g_\sigma\bar{\psi}\sigma\psi \;-\; g_\omega\bar{\psi}\not{\omega}\psi \;-\; g_\rho\bar{\psi}\not{\vec{\rho}}\vec{\tau}\psi - \; e\bar{\psi}\not{A}\psi\end{aligned} \tag{1}$$

The meson fields are the isoscalar σ meson, the isoscalar-vector ω meson and the isovector-vector ρ meson. The latter provides the necessary isospin asymmetry. The arrows denote the isovector quantities. The model contains also a non-linear scalar self-interaction of the σ meson :

$$U(\sigma) \;= \frac{1}{2}m_\sigma^2\sigma^2 \;+\; \frac{1}{3}g_2\sigma^3 \;+\; \frac{1}{4}g_3\sigma^4 \tag{2}$$

The scalar potential (2) introduced by Boguta and Bodmer [3] is essential for appropriate description of surface properties. M, m_σ, m_ω and m_ρ are the nucleon-, the σ-, the ω- and the ρ-meson masses respectively, while g_σ, g_ω, g_ρ and $e^2/4\pi = 1/137$ are the corresponding coupling constants for the mesons and the photon.

The field tensors of the vector mesons and of the electromagnetic field take the following form:

$$\begin{aligned}\Omega^{\mu\nu} &= \partial^\mu\omega^\nu - \partial^\nu\omega^\mu \\ \vec{R}^{\mu\nu} &= \partial^\mu\vec{\rho}^\nu - \partial^\nu\vec{\rho}^\mu \\ F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu\end{aligned} \tag{3}$$

The classical variational principle gives the equations of motion. In our approach, where the time reversal and charge conservation is considered, the Dirac equation

is written as:

$$\{-i\alpha\nabla + V(\mathbf{r}) + \beta[M + S(\mathbf{r})]\}\psi_i \;=\; \varepsilon_i\psi_i, \tag{4}$$

where $V(\mathbf{r})$ represents the vector potential:

$$V(\mathbf{r}) = g_\omega\omega_0(\mathbf{r}) + g_\rho\tau_3\rho_0(\mathbf{r}) + e\frac{1+\tau_3}{2}A_0(\mathbf{r}), \tag{5}$$

and $S(\mathbf{r})$ is the *scalar* potential:

$$S(\mathbf{r}) = g_\sigma\sigma(\mathbf{r}) \tag{6}$$

the latter contributes to the effective mass as:

$$M^*(\mathbf{r}) = M + S(\mathbf{r}). \tag{7}$$

The Klein-Gordon equations for the meson fields are time-independent inhomogeneous equations with the nucleon densities as sources.

$$\begin{aligned}
\{-\Delta + m_\sigma^2\}\sigma(\mathbf{r}) &= -g_\sigma\rho_s(\mathbf{r}) - g_2\sigma^2(\mathbf{r}) - g_3\sigma^3(\mathbf{r}) && (8)\\
\{-\Delta + m_\omega^2\}\omega_0(\mathbf{r}) &= g_\omega\rho_v(\mathbf{r}) && (9)\\
\{-\Delta + m_\rho^2\}\rho_0(\mathbf{r}) &= g_\rho\rho_3(\mathbf{r}) && (10)\\
-\Delta A_0(\mathbf{r}) &= e\rho_c(\mathbf{r}) && (11)
\end{aligned}$$

The corresponding source densities are

$$\begin{aligned}
\rho_s &= \sum_{i=1}^{A} \bar{\psi}_i\ \psi_i\\
\rho_v &= \sum_{i=1}^{A} \psi_i^+\ \psi_i\\
\rho_3 &= \sum_{p=1}^{Z} \psi_p^+\ \psi_p\ -\ \sum_{n=1}^{N} \psi_n^+\ \psi_n\\
\rho_c &= \sum_{p=1}^{Z} \psi_p^+\ \psi_p
\end{aligned} \tag{12}$$

where the sums are taken over the valence nucleons only. It should also be noted that the present approach neglects the contributions of negative-energy states ($no-sea$ approximation), i.e. the vacuum is not polarized.

3 NUMERICAL DETAILS

The Dirac equation for nucleons is solved using the method of oscillator expansion as described in Ref. [4]. For determination of the basis wavefunctions an axially symmetric harmonic-oscillator potential with size parameters

$$b_z = b_z(b_0, \beta_0) = b_0 \exp(\sqrt{5/(16\pi)}\beta_0) \quad (13)$$

$$b_\perp = b_\perp(b_0, \beta_0) = b_0 \exp(-\sqrt{5/(64\pi)}\beta_0) \quad (14)$$

is employed. The basis is defined in terms of the oscillator parameter b_0 and the deformation parameter β_0. The oscillator parameter b_0 is chosen as $b_0 = 41A^{-1/3}$ and the basis deformation β_0 is determined for each nucleus in such a way that the resulting mass quadrupole moment Q of the nucleus is given by $Q = \sqrt{\frac{16\pi}{5}}\frac{3}{4\pi}AR_0^2\beta_0$ with $R_0 = 1.2A^{1/3}$.

For, most of the nuclei considered here are open-shell nuclei both in protons and neutrons (except those with N=82), pairing has been included. We have used the BCS formalism for the pairing as in our previous works [4]. Constant pairing gaps have been used, which are taken from the empirical particle separation energies of neighboring nuclei and in cases where empirical data are not known, the gaps have been extrapolated according to the empirical rule $\Delta = 11.2/\sqrt{N(Z)}$ (MeV). The zero-point energy of an harmonic oscillator has been used for the center-of-mass energy correction. We have neglected the angular momentum and particle number projection as well as the ground state correlations induced by coupling to collective vibrations. It is, however, expected that these additional

corrections would have only small contributions. Here, therefore, we aim to describe the ground-state properties of nuclei within the realm of the pure mean field.

The number of oscillator shells taken into account is 12 for both fermionic and bosonic wavefunctions. For convergence reasons we also considered 14 shells for a trial. It turned out, however, that the difference in the results of 12 and 14 shells is negligible. All the calculations were hence performed in a basis of 12 harmonic oscillator shells.

The parameter set NL1 has been shown to provide reasonably good results for nuclei about the line of stability [4]. However, because of a very large asymmetry energy 44 MeV of NL1 as compared to the empirical value, NL1 does not provide a good description of nuclei away from the stability line. This has also the consequence that it overestimates the neutron-skin thickness [45] of nuclei with large neutron excess. This problem has been remedied in the force NL-SH, whereby the ρ-meson coupling constant and thus the asymmetry energy has been brought very close to the empirical value. In this paper we have therefore used the force NL-SH. This force has subsequently been shown to provide excellent results [5, 8] for nuclei on both the sides of the stability line. The parameters of NL-SH have been taken from Ref. [5, 7]. The exact values and units of parameters of the force NL-SH are:

M = 939.0 MeV; m_σ = 526.059 MeV; m_ω = 783.0 MeV; m_ρ = 763.0 MeV;

$g_\sigma = 10.444$; $g_\omega = 12.945$; $g_\rho = 4.383$; $g_2 = -6.9099$ fm^{-1} ; $g_3 = -15.8337$.

Charge densities and the corresponding charge radii are obtained by folding the proton point densities with the proton form factor of the Gaussian shape. This leads, in a rather good approximation, to the formula

$$r_c \ = \ \sqrt{r_p^2 + 0.64}, \tag{15}$$

where the rms-radius of the proton has been taken to be 0.8 fm.

The quadrupole moments for protons and neutron are calculated according to the usual definition [46]

$$Q_2 = \langle 2r^2 P_2(\cos\theta)\rangle = \langle 2z^2 - x^2 - y^2\rangle \quad (16)$$

$$Q_4 = \langle r^4 Y_{40}(\theta)\rangle = \sqrt{\frac{9}{4\pi}}\frac{1}{8}\langle 8z^4 - 24z^2(x^2+y^2) + 3(x^2+y^2)^2\rangle \quad (17)$$

The quadrupole deformation parameter β_2 and the hexadecapole deformation parameter β_4 are obtained in such a way, that sharp edged densities with this deformations have the same multipole moments, as discusses in the appendix of Ref. [46].

4 RESULTS AND DISCUSSION

We have performed relativistic Hartree calculations for six isotopic chains in the rare-earth region. Isotopes of Nd (Z=60), Sm (Z=62), Gd (Z=64), Dy (Z=66), Er (Z=68) and Yb (Z=70) have been considered. For, many nuclei in these chains are known to be well deformed and several shape transitions along these isotopic chains are expected, we have solved the RMF equations for an axially deformed configuration both for prolate as well oblate shape. However, we present the results only for the lowest energy shape. Since we do not include triaxial degrees of freedom we cannot decide whether the second minimum is a local minimum or a saddle point. On the other hand, empirically only absolute values of the quadrupole deformations are known. Some microscopic calculations using the Gogny force and Skyrme force SIII [23] do predict the shapes of a few nuclei. However, extensive microscopic predictions on the exact shapes are lacking. In this context, the empirical isotope shifts which are sensitive to the size and the sign of deformation serve to reveal the character (signature) of the shapes (Ref.

[16]).

Calculations for Nd isotopes include mass numbers A=130 up to A=162, for Sm isotopes A=134 up to A=164, for Dy isotopes A=142 up to A=168, while the Er and Yb isotopes cover the region from A=142 to A=170 and A=154 to A=184, respectively. On the chains of Nd, Sm, Dy, Er and Yb several precision measurements [16] on the isotope shifts are available. We have also performed calculations for Gd nuclei in the region $136 \leq A \leq 172$. We compare our results on this chain with some experimental data derived from Ref. [34], where measurements on mean-square charge radii of some Gd nuclei have been performed.

4.1 Binding Energies

In Tables 1-6 we present the total binding energies of the isotopes considered in this work together with the predictions of the mass formulae Finite Range Droplet Model (FRDM) [47] and Extended Thomas-Fermi and Strutinsky Integral (ETF-SI) [48]. The last column of the tables shows the available empirical values from the recently published atomic mass evaluation [49]. The calculated binding energies in the RMF theory are in good agreement with the empirical data for all the isotopic chains and thus the binding energies of a large number of nuclei have been reproduced by the RMF theory. For only a few nuclei full agreement is not obtained. However the deviations of the RMF values from the empirical ones is at the most 0.3%. Our results are also in overall good agreement with the values of both the mass models which stretch over all the isotopic chains and the mass ranges considered here. Only for a few neutron-deficient Er nuclei is the disagreement with the models slightly higher than for other isotopes, and is about 0.4%. The agreement of the RMF theory with the empirical data and with the mass models is here noteworthy, especially as non-linear the RMF theory entails

only 6 parameters fitted to six spherical nuclei at the stability line. It is worth noting that in contrast both the mass models FRDM and ETF-SI have been fitted exhaustively to varying number of parameters to include empirical data of over one thousand known nuclei. It is, therefore, gratifying that the RMF theory with only 6 parameters is able to reproduce the general trend of the absolute binding energies within an rms deviation of about 2 MeV.

We show in figures 1-2 the neutron and proton single-particle spectra near the Fermi energy (indicated by dashed line) for the nucleus ^{166}Er as a characteristic case. A comparison is made with the single-particle spectra taken from from the density-dependent Skyrme forces SIII and SkM [26]. The single-particle spectrum from the Modified Harmonic Oscillator [50] (MHO) is also shown. The levels are labeled using the standard notation Ω^{π}. The nucleus ^{166}Er is highly deformed in its ground state with $\beta_2 = 0.34$. A comparison of single proton spectra from the RMF and Skyrme theories shows that the RMF parameter set NL-SH has larger gaps between levels. The Skyrme force SkM also shows gaps which are slightly smaller than those in NL-SH. The level density of spectra in NL-SH and SkM are similar. In comparison, SIII shows more evenly distributed single particle levels than both NL-SH and SkM. On the other hand, the MHO spectrum is denser than both the RMF and Skyrme forces. It is interesting to note that though both NL-SH and SIII have a larger compressibility modulus of about $K = 355$ MeV, this fact does not reflect itself in the single-particle spectra.

The neutron single-particle spectra shown in figure 2 also look similar to those of protons except that SkM shows lesser gaps for neutrons than for protons. The NL-SH spectrum has clearly higher gaps for the neutrons than the corresponding spectrum of SkM. On the other hand, the SIII spectrum is slightly denser than both NL-SH and SkM. The MHO spectrum is, in this case, much denser than

both the RMF and Skyrme spectra. This can be understood by the effective mass $m^*/m = 1$ in MHO. The Fermi energy in all the cases is very close to each other. The difference in the spin-orbit interaction in the RMF theory and the Skyrme approach should lead to differences in details of the single-particle spectra and hence would also reflect in the magnitude of the nuclear radii.

4.2 Radii and Isotope shifts

The nuclear radii obtained in the RMF theory are shown in figures 3-8. Here the charge and neutron radii obtained for Nd, Sm, Gd, Dy, Er and Yb isotopes, respectively, have been provided. The neutron radii show an increasing trend with neutron number for all the isotopic chains. However, for Er nuclei it shows a clear kink about the magic neutron number N=82. A much weaker kink about N=82 is also visible in the neutron radii of Nd, Sm and Dy chains. This is an indication of the shell effects in these nuclei. Such shell effects leading to a kink have been observed in the empirical isotope shifts of Pb nuclei.

The neutron radii for Gd isotopes (Fig. 5) show an exceptional behaviour in the sense that for the lighter Gd nuclei and in particular 4-6 neutrons below the closed neutron shell, nuclei seem to acquire a neutron radius in unison with the corresponding charge radii, i.e. the lighter isotopes have a larger size than those of the heavier magic-neutron counterpart. This feature has been symptomatic of the empirical charge radii of Sr and Kr isotopic chains, the effect which has been described successfully in the RMF theory. This behaviour of the neutron radii of Gd nuclei can be put in perspective with the above feature assuming Z=64 shows some magicity.

The RMF charge radii have been used to obtain isotopes shifts $r_c^2(A) - r_c^2(\mathrm{ref})$ for all the isotopic chains, where $r_c(\mathrm{ref})$ denotes rms charge radius of the respec-

tive reference nuclei. The semi-magic nuclei ^{142}Nd, ^{144}Sm, ^{146}Gd, ^{148}Dy, ^{150}Er having closed neutron-shell with N=82 have been used as reference points. Only for the isotopic chain of Yb nuclei, the nucleus ^{168}Yb which is without magic neutron number, has been used as a reference. This has been done with a view to facilitate the comparison of our predictions with the empirical data as done by Otten [16]. In figures 9-14 we present our results for the isotopic shifts. The empirical data has also been shown wherever available.

The experimental isotope shifts for Nd, Sm, and Dy chains show a value close to zero or slightly positive for nuclei below N=82, and a pronounced kink in empirical curves can be seen. This tendency is very much similar to those of the empirical isotope shifts of Sr and Kr chains [6]. For the Er and Yb chain, data below N=82 is not available. It can be seen from figures 9, 10, and 12 that the empirical data on the Nd, Sm and Dy isotopes on both the sides of the neutron closed-shell can be very well reproduced in the RMF theory using the parameter set NL-SH. The kink in the isotope shifts of our calculations appears for all nuclei except for Yb where we have not included the semi-magic nucleus due to the unavailability of empirical data. The kink and the structure about and below the semi-magic nuclei N=82 stems primarily from the onset of deformation which tend to overweigh the rms charge radii of the lighter nuclei with respect to the corresponding magic neutron nuclei. The RMF theory describes this behaviour very well and predicts a similar behaviour also for nuclei for which data do not exist.

For Gd nuclei (Fig. 11) the isotope shifts for the very light isotopes vary slightly differently than the other chains in the RMF theory. The isotopic shift for the nucleus ^{144}Gd (N=80) is a little negative and then for even lighter isotopes, the isotopic shifts increase with a decrease in neutron number down to A=140. This

implies that Gd nucleus with A=144 has the smallest rms charge distribution and that the nucleus ^{140}Gd has a much bigger rms charge radius than the semi-magic nucleus ^{146}Gd. This effect is clearly visible in Fig. 5. It can partly be attributed to a large and abrupt shape transition from prolate to oblate and again back to prolate in the region of A=140-144, a behaviour which will become clearly apparent in Fig. 19.

The experimental isotopic shifts on Gd nuclei have been measured by a Russian group [34]. The measurements pertain to the mean-square charge radii of several Gd isotopes with neutron numbers above N=82. We have derived the isotope shifts from this work, which have been shown in Fig. 11 for comparison. The empirical isotopic shifts coincide practically with the predictions of the RMF theory.

For the Yb chain, we have shown the results by taking ^{168}Yb as a reference nucleus. We observe that the RMF values lie close to the empirical values on both the sides of the reference nucleus. An overall view of the isotope shifts of all the chains except Yb shows that the behaviour of the charge radii and thus of the isotope shifts is consistently similar. It can be noted that the RMF theory is able to reproduce the available data very well.

Changes in charge radii with respect to neighboring even-even nuclei can be best reflected by the so-called Brix-Kopferman plot. In Fig. 15 we show the differential changes of the mean-square charge radii of the neighboring isotopes, $\delta < r^2 >^{N-2,N}$, in the Brix-Kopfermann diagram [51, 52]) obtained from the RMF theory. A strong peak at N=90 can be seen clearly in this quantity for the isotopic chains of Nd, Sm and Gd. This differential change decreases considerably above N=90 and then flattens. The peak at N=90 corresponds to a sudden onset of strong static deformation for nuclei below N=90 in these isotopic chains. A

small value of the differential change in the mean-square charge radius above N=94 depicts a saturation in the value of the quadrupole deformations of nuclei. This effect will become clear from the magnitude of deformations in the next figures. It is also to be seen in Fig. 15 that this peak gradually diminishes going from Nd to Yb. An additional small peak also appears at N=86 for some nuclei such as Gd and Dy. For the sake of comparison with the experimental curve, we show in Fig. 16 the Brix-Kopferman plot derived from the empirical isotope shifts [16]. The empirical Brix-Kopferman diagram shows a structure similar to that obtained in the RMF theory. The main peak appears at N=90 as predicted in the RMF theory. Thus, the RMF theory agrees with the empirical data very well. This fact has already been seen in the isotope shifts in figures 9-14. However, a secondary peak in the RMF curve appearing at N=86 is not to be seen in the empirical data. Instead a small peak appears at N=84 in the empirical curve. This discrepancy in the plots about N=84-86 can be attributed to a complex and slightly different evolution of the shapes and the magnitudes of deformation on adding neutrons to the closed-shell (N=82) nuclei.

4.3 Deformations Properties

4.3.1 Quadrupole deformation

The deformations and shapes of nuclei play a crucial role in defining the properties such as nuclear sizes and isotope shifts. In the RMF theory we have obtained the quadrupole and hexadecapole moments of nuclei from the solution of deformed RMF equations. The resulting quadrupole and hexadecapole deformation parameters β_2 and β_4 are calculated using the method of Ref. [46] as discussed at the end of Section III.

The β_2 values are shown in Figs. 17-22 for all the isotopic chains considered in

this work. Since the mass formulae FRDM and ETF-SI provide the deformations obtained from their exhaustive fits, we supplement the figures with the predictions of these models for the sake of comparison. As the empirical values of β_2 obtained from BE(2) values do not contain the sign of the deformation, we do not show the empirical values in these figures. However, the empirical values will be given in the tables below.

The β_2 values obtained in the RMF theory manifest an interesting change of shapes of nuclei below and above the magic neutron number N=82. Interestingly, we find that for most of the chains, there is an excellent agreement of the RMF predictions with the mass models. In greater number of cases the RMF values are much closer to the FRDM than the ETF-SI both in the variation (including shape transitions) as well as in the magnitude. For nuclei such as Nd and Sm (Figs. 17 and 18), the ETF-SI model predicts a slightly different variation of deformation with mass number than that predicted by the RMF theory and the FRDM.

For most of the chains, there is an onset of prolate deformation above N=82. All the nuclei with N=82 are spherical as expected. However, a successive addition of neutrons to the magic neutron core leads to a gradual evolution of prolate shape in the RMF theory. The nucleus ^{148}Gd is an exception where the RMF theory predicts a slightly oblate shape. For neutron numbers higher than N=82, the prolate deformation increases and then saturates at a value close to $\beta_2 = 0.30-0.35$ in most of the cases. In comparison, both the FRDM and ETF-SI also predict an increasing prolate deformation for nuclei above N=82. The ETF-SI predicts an early onset of the prolate deformation as compared with NL-SH and FRDM for most of the chains considered here. The quadrupole deformation values in the ETF-SI for nuclei immediately closer to N=82 are higher than both

NL-SH and the FRDM. This behaviour can be seen in Figs. 18-21. In general, there is a striking similarity in the predictions of the RMF theory and the FRDM and ETF-SI results for nuclei above N=82.

For nuclei below N=82, all the isotopic chains (except Yb which does not encompass N=82) exhibit an almost identical behaviour in the quadrupole deformation. Nuclei in all the chains (Figs. 17-21) undergo a spherical to oblate shape transition below the neutron magic number except for Gd (Fig. 19) and Dy (Fig. 20) nuclei, where nuclei just below magic number (N=80) acquire a prolate shape in the RMF theory. The sharp transition to the oblate shape in the five chains is taking place at N=78 in the RMF theory. Only in the case of Er this transition occurs smoothly from a spherical shape via a slightly less oblate shape for ^{148}Er (N=80). All nuclei below N=78 suddenly acquire a prolate shape the magnitude of which increases for Nd and Sm isotopes with a decrease in the neutron number. It is worth noting that the oblate shape at N=78 for all chains, the prolate shape at N=80 for Gd and Dy and the shape transition from oblate to prolate in going to lower neutron numbers below N=78 are the aspects which are meticulously consistent in the RMF theory and in the FRDM for all the above five chains. The ETF-SI does show these features for Gd, Dy and Er (Figs. 19-21) and is consistent both with NL-SH as well as FRDM in these complex series of transitions. However, the ETF-SI tends to predict a slightly higher prolate deformation for the above nuclei than that predicted by NL-SH and FRDM. For the isotopic chains of Nd and Sm (Figs. 17-18) and for nuclei below N=82, the ETF-SI values deviate strongly from the predictions of NL-SH and FRDM. For Nd (Z=60) isotopes ETF-SI predicts an increasingly prolate shape for nuclei below and including N=82, without giving a spherical shape to the neutron magic nucleus ^{142}Nd. The NL-SH and FRDM provide a spherical shape

to nuclei two neutron numbers below and above N=82 for Nd and Sm. Thus, in the five isotopic chains (excluding Yb) the RMF theory as well as FRDM predict a complex series of prolate-oblate-spherical-prolate shape transitions with increasing neutron number.

The chain of Yb (Z=70) isotopes does not include a magic neutron number. The quadrupole deformations for nuclei above N=84 (Fig. 22) show an increasingly prolate shape with an increase in neutron number in the RMF theory as well as in FRDM and ETF-SI. The β_2 value saturates above N=98 in all the three approaches. The magnitude of the β_2 value in NL-SH is in some cases closer to FRDM and in others closer to ETF-SI. The general trend for Yb isotopes is similar in all the three approaches.

The numerical values of the quadrupole deformations in the RMF theory are given in Tables 7-12. The β_2 values from the FRDM and ETF-SI are also provided in the tables for the sake of comparison of magnitude and signature. The absolute values of the empirical β_2 obtained from the compilation of Raman et al. [53] are shown in the last columns. It must be stressed that these values do not indicate the sign of the quadrupole moment. For the nuclei where the experimental value of the deformation has been deduced, the β_2 values from NL-SH correspond closely to the empirical ones in magnitude. For some specific nuclei such as ^{152}Sm and ^{154}Sm, the β_2 values obtained by fitting the differential cross-sections in electron scattering experiments [35] have been given in the last column of Table 8 in the parentheses. The RMF values show a significant agreement with these empirical values. Thus, the RMF theory describes all the available data on the deformation of the rare-earth nuclei successfully. The detailed behaviour of shapes and shape transitions has already been alluded to above.

A comparison of the NL-SH values and the empirical quadrupole deformation

has been facilitated in Fig. 23, where we show β_2^2 for all the isotopic chains. The figure shows an overall consistency of the theoretical values with the empirical ones. However, in presenting the squares of β_2 any slight discrepancy is likely to be amplified. Taking this fact into account, there is a broad agreement between the RMF deformations and the empirical values. This is corroborated by our Tables 7-12.

In order to judge the comparative behaviour of various isotopic chains towards the quadrupole deformation, we show the loci of $< \beta_2^2 >$ in Fig. 24, as a function the neutron number. The minima at the magic number N=82 are clearly visible. On both the sides of this number, a parabola like behaviour is to be seen. Nuclei such as Nd, Sm and Gd exhibit strong shape transition and highly deformed shapes above N=88. They reach a saturation at about N=102. For the other chains such as Dy, Er and Yb, this trend is much more gradual. For Yb isotopes the maximum deformation is a little less than those of its neighboring chains.

4.3.2 Hexadecapole deformations

The hexadecapole deformation has usually been inferred by fitting cross-sections in inelastic scattering experiments [54] and from experiments with Coulomb excitation [36]. Empirical data on the hexadecapole deformations of nuclei is available only scantily. On the other hand, most of the mass formulae such as FRDM and ETF-SI employ a hexadecapole degree of freedom in the minimization in their exhaustive and global fits. The FRDM and the ETF-SI thus predict the hexadecapole deformation β_4 along with the quadrupole deformation β_2 which has been discussed above.

First, we show the hexadecapole deformation β_4 obtained with NL-SH for all the isotopic chains in Tables 13-18. Predictions of the FRDM and ETF-SI

are also shown for comparison. For all the isotopic chains except Yb (which does not include a magic neutron number), nuclei below N=82 exhibit a negative value of β_4. Comparing this with the sign of the β_2, this is in contrast with the overwhelmingly positive β_2 values for neutron numbers in this region, except for the oblate shape at N=78. The β_4 values then edge to zero at and around the magic numbers where nuclei are expected to be spherical. This is consistent with the corresponding quadrupole deformation β_2 which vanishes about N=82. The FRDM also predicts negative β_4 values below N=82. This is again in consistency with the RMF predictions. The ETF-SI, on the other hand, is consistent with RMF and FRDM for Gd, Dy and Er isotopes. In contrast to RMF and FRDM, ETF-SI gives positive β_4 values for the Nd and Sm nuclei below N=82. A comparison of the theoretical predictions with the empirical hexadecapole moments will be made below, wherever experimental data are available.

For nuclei above N=82, the RMF theory predicts a positive β_4 for all the chains except Yb (Table 18). The corresponding quadrupole deformation for all the chains including Yb have a prolate shape. Only for the Yb chain, the β_4 values are negative in clear contrast with all other chains, though all the Yb isotopes above N=82 show a prolate quadrupole shape. Again, the predictions of the RMF theory on the sign of the hexadecapole deformation are consistent with those of the FRDM and ETF-SI for all the chains including that of Yb.

In Tables 19-24 we show our predictions on the charge hexadecapole moment calculated with the force NL-SH. The experimental data from a very recent compilation by Löbner [55] are also shown where available. It may be noted that the same convention for the empirical data is used. In some cases more than one values is provided as having been obtained using different probes. In many cases, where more than one empirical value for a nucleus is given, there is a

unanimity in the values. In the others the empirical values indicate a rather broad differences and sometimes the error bars are too large to reach a conclusion on the value. However, in general, there is a broad and good agreement of the RMF hexadecapole moments with the empirical data notwithstanding the inherent uncertainties which some data possess.

5 Summary and Conclusions

The relativistic mean-field theory with the non-linear $\sigma\omega$ model has been employed to investigate the ground-state properties of several chains of rare-earth nuclei. The calculations have been performed for deformed axially symmetric configurations in the relativistic mean-field approximation for the even-even nuclei of the isotopic chains of Nd, Sm, Gd, Dy, Er and Yb. Binding energies, isotopic shifts and deformation properties have been obtained using the parameter set NL-SH. The results of these calculations have been compared with the empirical data available on the binding energies, isotopic shifts, and quadrupole and hexadecapole deformations.

The RMF theory describes the binding energies of nuclei over a large range of proton and neutron numbers very well. The RMF binding energies are also in good agreement with the predictions of the extensive mass formulae FRDM and ETF-SI. The empirical isotopic shifts of all the isotopic chains have been described successfully. It is noteworthy that the behaviour of the isotopic shifts in the deformed rare-earth nuclei is reminiscent of those of the Sr and Kr isotopic chains, the data which have earlier been reproduced [6] well only within the RMF theory.

The quadrupole deformations predicted by the RMF theory have been found to be in very good agreement with the empirical data. The magnitude of the

quadrupole deformation β_2 in the RMF theory shows a good agreement also with those of FRDM and ETF-SI. In addition, the RMF theory predicts a complex web of shape transitions, which are observed to be similar for most of the isotopic chains. The rare-earth nuclei in this region exhibit shape transitions prolate-oblate-spherical-prolate for all the chains (except Yb) with an increasing neutron number. The shape transition from prolate to oblate at N=78 is spectacular as nuclei on both the sides of N=78 assume deformations in the direction of positive β_2. The complex series of shape transitions both below and above the neutron magic number N=82 are in astonishing agreement with the FRDM. With some exceptions, the RMF results also show good agreement with ETF-SI.

The hexadecapole moments and β_4 values obtained in the RMF theory show a good comparison with the corresponding empirical values wherever available. The RMF theory provides a negative β_4 values for nuclei below the neutron magic number N=82. In contrast, the β_4 values for nuclei above N=82 are positive for all the chains. For the case of Yb chain only, nuclei above N=100 exhibit negative β_4 values. Comparing with the mass models, a very good agreement of the RMF predictions on the β_4 values has been observed with the predictions of the FRDM for all the isotopic chains. Again, with a few exceptions only, the RMF values are in accord with the ETF-SI predictions.

It may be reckoned [6] that the RMF theory with the force NL-SH has been successful in providing a good description of the anomalous isotopic shifts of Sr and Kr nuclei which undergo a series of complex shape transitions as also is the case with the rare-earth nuclei. The present calculations on the rare-earth nuclei in conjunction with the results of Sr and Kr nuclei demonstrate the ability of the RMF theory with the force NL-SH to describe a broad range of nuclear data encompassing a large range of nuclei and isospins.

In retrospect, calculations using non-relativistic Skyrme and Gogny forces were carried out for deformed nuclei by several groups. Extensive calculations of entire chains of isotopes with these approaches are not available. On the other hand, some of these approaches using the Skyrme force SIII and the Gogny force D1 have been able to reproduce results on deformation properties such as quadrupole and hexadecapole moments and on binding energies for a set of selected nuclei. However, these approaches have not been able to provide an adequate description of the anomalous nature of the isotopic shifts in deformed rare-earth nuclei [16]. Also, it is not clear how these approaches will respond to the properties over a large range of isospin.

A correct description of the isotopic shifts depends crucially upon the deformations assumed by nuclei. The ability of the RMF theory in describing the isotopic shifts of rare-earth nuclei as well as those of Sr and Kr nuclei stems from the successful description of the deformations. In this context, the shell effects play a major role in determining the potential energy landscape and consequently the deformation of the ground state. It has been pointed out [14] that shell effects in the RMF theory are different from those in the non-relativistic approaches. Thus, the appropriate shell effects render a unified and comprehensive description of several aspects of the ground-state properties of nuclei.

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Figure Captions

Fig. 1 The neutron single-particle (s.p.) spectrum for ^{166}Er near Fermi energy obtained in the RMF theory using the force NL-SH. The spectra from the Skyrme forces SIII and SkM are shown for comparison. The corresponding spectrum using the Modified Harmonic Oscillator (MHO) is also given.

Fig. 2 The proton s.p. energies for ^{166}Er. For details, see the caption of Fig. 1.

Fig. 3. The charge and neutron radii of Nd isotopes in the RMF theory. For details, see text.

Fig. 4 The charge and neutron radii of Sm isotopes in the RMF theory. For details, see text.

Fig. 5 The charge and neutron radii of Gd isotopes in the RMF theory. For details, see text.

Fig. 6 The charge and neutron radii of Dy isotopes in the RMF theory. For details, see text.

Fig. 7 The charge and neutron radii of Er isotopes in the RMF theory. For details, see text.

Fig. 8 The charge and neutron radii of Yb isotopes in the RMF theory. For details, see text.

Fig. 9 The isotope shifts for Nd nuclei obtained in the RMF theory with the force NL-SH. The empirical data [16] are also shown for comparison. A kink about the N=82 nucleus can been clearly.

Fig. 10 The same as in Fig. 9, for Sm nuclei.

Fig. 11 The same as in Fig. 9. However, the empirical data has been obtained from Ref. [34].

Fig. 12 The same as in Fig. 9, for Dy nuclei.

Fig. 13 The same as in Fig. 9, for Er nuclei.

Fig. 14 The isotope shifts for Yb nuclei. The nucleus ^{168}Yb has been used as a reference point.

Fig. 15 The Brix-Kopfermann plot obtained in the RMF theory with the force NL-SH.

Fig. 16 The Brix-Kopferman plot using the experimental isotope shifts.

Fig. 17 The quadrupole deformation β_2 for Nd isotopes with NL-SH. The predictions of the mass models FRDM and ETF-SI are also shown for comparison. See text for details.

Fig. 18 The same as in Fig. 17, for Sm isotopes.

Fig. 19 The same as in Fig. 17, for Gd isotopes.

Fig. 20 The same as in Fig. 17, for Dy isotopes.

Fig. 21 The same as in Fig. 17, for Er isotopes.

Fig. 22 The same as in Fig. 17, for Yb isotopes.

Fig. 23 β_2^2 for various rare-earth nuclei in the RMF theory (open circles). The available experimental values (solid circles) [53] from BE(2) measurements are also shown for comparison. For details see text.

Fig. 24 Loci of β_2^2 obtained in the RMF theory, for various isotopic chains of rare-earth nuclei. A parabolic behaviour about the magic neutron number is observed.

Table 1: The binding energies (in MeV) for Nd isotopes obtained with the force NL-SH. The predictions from the mass models FRDM and ETF-SI are also shown for comparison. The empirical values (expt.) are shown in the last column.

A	NL-SH	FRDM	ETF-SI	expt.
130	1070.39	1069.10	1068.82	1068.67
132	1091.64	1089.97	1090.16	1090.09
134	1112.01	1109.75	1109.86	1110.38
136	1131.39	1129.09	1128.92	1129.92
138	1151.42	1147.92	1148.37	1148.94
140	1172.87	1167.15	1167.27	1167.52
142	1190.19	1185.67	1185.85	1185.15
144	1203.39	1199.16	1199.28	1199.09
146	1214.46	1212.00	1212.52	1212.41
148	1226.05	1225.33	1225.56	1225.03
150	1238.39	1238.46	1238.35	1237.45
152	1251.07	1250.93	1250.48	1250.06
154	1262.03	1262.45	1261.72	1261.66
156	1272.74	1273.01	1272.33	1272.29
158	1282.36	1282.88	1282.21	-
160	1292.42	1292.14	1291.24	-
162	1300.43	1300.81	1299.33	-

Table 2: The binding energy (in MeV) of Sm isotopes. For details refer to the caption of Table 1

A	NL-SH	FRDM	ETF-SI	expt.
134	1097.82	1094.65	1095.16	1094.51
136	1118.91	1115.72	1116.09	1115.98
138	1138.85	1136.17	1135.89	1136.56
140	1159.90	1156.00	1156.09	1156.47
142	1180.37	1176.29	1176.31	1176.61
144	1200.91	1196.09	1196.27	1195.74
146	1213.98	1210.82	1210.91	1210.91
148	1227.99	1225.00	1225.67	1225.40
150	1241.40	1239.67	1240.23	1239.25
152	1255.75	1254.11	1254.22	1253.12
154	1269.60	1267.87	1267.83	1266.94
156	1281.94	1280.72	1280.39	1279.99
158	1294.40	1292.59	1291.91	1292.03
160	1305.34	1303.86	1303.35	1303.19
162	1315.35	1314.42	1313.48	-
164	1324.63	1324.25	1322.98	-

Table 3: The binding energies for Gd isotopes. See the caption of Table 1 for details.

A	NL-SH	FRDM	ETF-SI	expt.
138	1123.32	1119.77	1119.82	1119.69
140	1144.41	1141.48	1141.12	1141.70
142	1165.99	1162.27	1162.30	1163.48
144	1183.84	1183.54	1183.37	1183.55
146	1207.42	1204.45	1204.73	1204.44
148	1222.86	1220.77	1220.50	1220.76
150	1239.31	1236.07	1236.74	1236.40
152	1254.51	1251.88	1252.68	1251.49
154	1269.44	1267.38	1267.95	1266.63
156	1284.27	1282.24	1282.56	1281.60
158	1298.31	1296.23	1296.57	1295.90
160	1311.72	1309.47	1309.38	1309.29
162	1323.80	1322.00	1321.50	1321.77
164	1335.02	1333.81	1333.52	1333.37
166	1345.94	1344.81	1344.09	-

Table 4: The binding energies for Dy isotopes. See the caption of Table 1 for details.

A	NL-SH	FRDM	ETF-SI	expt.
142	1148.15	1144.78	1144.30	1144.42
144	1170.83	1166.67	1166.91	1167.39
146	1191.24	1189.00	1188.62	1189.64
148	1213.60	1211.24	1211.22	1210.83
150	1230.70	1228.50	1228.20	1228.40
152	1248.99	1245.14	1245.67	1245.33
154	1265.66	1262.08	1262.88	1261.75
156	1281.93	1278.61	1279.40	1278.04
158	1297.73	1294.55	1295.24	1294.06
160	1312.94	1309.69	1310.30	1309.47
162	1327.17	1324.17	1324.56	1324.12
164	1340.21	1337.98	1337.98	1338.05
166	1352.99	1350.99	1350.60	1350.81
168	1365.63	1263.26	1363.11	1362.82

Table 5: The binding energies for Er isotopes. See the caption of Table 1 for details.

A	NL-SH	FRDM	ETF-SI	expt.
142	1127.36	1121.81	1121.16	-
144	1150.42	1146.00	1144.91	-
146	1173.84	1169.06	1169.02	1169.97
148	1197.49	1192.59	1192.31	1193.11
150	1219.12	1215.78	1215.84	1216.15
152	1237.40	1234.32	1234.00	1234.19
154	1257.26	1252.25	1252.54	1252.40
156	1275.45	1270.15	1270.96	1270.39
158	1288.95	1287.64	1288.67	1286.47
160	1307.23	1304.70	1305.68	1304.27
162	1324.00	1321.00	1321.95	1320.70
164	1339.53	1336.69	1337.31	1336.45
166	1353.29	1351.75	1352.17	1351.57
168	1366.65	1366.03	1366.05	1365.78
170	1379.93	1379.51	1379.09	1379.78

Table 6: The binding energies for Yb isotopes. See the caption of Table 1 for details.

A	NL-SH	FRDM	ETF-SI	expt.
154	1241.63	1238.36	1238.12	1238.97
156	1262.73	1257.42	1257.48	1257.67
158	1281.68	1276.32	1276.87	1276.53
160	1298.55	1294.80	1295.70	1294.81
162	1315.78	1312.70	1313.84	1312.64
164	1333.89	1330.19	1331.26	1329.93
166	1350.45	1347.03	1347.84	1346.67
168	1365.36	1363.22	1363.71	1362.79
170	1379.96	1378.66	1379.05	1378.13
172	1394.22	1393.31	1393.27	1392.77
174	1408.23	1407.00	1406.95	1406.60
176	1420.62	1420.09	1419.76	1419.29
178	1433.41	1432.78	1431.31	1431.63
180	1443.76	1443.69	1442.41	-
182	1453.79	1454.33	1452.92	-
184	1462.70	1464.34	1462.63	-

Table 7: The quadrupole deformations β_2 for Nd isotopes obtained in the RMF theory using the force NL-SH. The FRDM and ETF-SI predictions are also shown. The available empirical deformations (expt.) obtained from BE(2) values are also given in the last column. The experimental values do not depict the sign of the deformation.

A	NL-SH	FRDM	ETF-SI	expt.
130	0.322	0.311	0.36	-
132	0.268	0.293	0.37	-
134	0.229	0.218	0.36	-
136	0.182	0.171	0.36	-
138	-0.091	-0.138	0.19	-
140	0.000	0.000	0.15	-
142	0.000	0.000	0.08	0.093
144	0.000	0.000	0.08	0.131
146	0.080	0.161	0.14	0.152
148	0.118	0.206	0.21	0.204
150	0.264	0.243	0.24	0.289
152	0.318	0.262	0.29	0.274
154	0.331	0.270	0.31	-
156	0.338	0.279	0.31	-
158	0.345	0.279	0.31	-
160	0.349	0.290	0.32	-
162	0.350	0.300	0.32	-

Table 8: The quadrupole deformations β_2 for Sm isotopes. See Table 7 for details. The empirical values given in the parentheses in the last column are from electron scattering experiments of Ref.[35]

A	NL-SH	FRDM	ETF-SI	expt.
134	0.301	0.312	0.370	-
136	0.263	0.237	0.360	-
138	0.214	0.190	0.270	0.225
140	-0.124	-0.148	0.150	-
142	0.0	0.00	0.09	-
144	0.0	0.00	0.00	0.088
146	0.0	0.0	0.120	-
148	0.108	0.161	0.200	0.142
150	0.166	0.206	0.230	0.193
152	0.261	0.243	0.260	0.306 (0.287 ± 0.003)[35]
154	0.309	0.243	0.280	0.341 (0.311 ± 0.003)[35]
156	0.324	0.279	0.300	-
158	0.335	0.279	0.310	-
160	0.344	0.290	0.310	-
162	0.349	0.300	0.320	-
164	0.350	0.320	0.340	-

Table 9: The quadrupole deformations β_2 for Gd isotopes. See Table 7 for details.

A	NL-SH	FRDM	ETF-SI	expt.
138	0.295	0.256	0.37	-
140	0.307	0.210	0.33	-
142	-0.158	-0.156	-0.21	-
144	0.081	0.000	0.06	-
146	0.000	0.000	0.0	-
148	-0.063	0.000	0.12	-
150	0.140	0.161	0.20	-
152	0.185	0.207	0.25	0.212
154	0.264	0.243	0.27	0.310
156	0.314	0.271	0.30	0.338
158	0.330	0.271	0.30	0.348
160	0.343	0.280	0.31	0.353
162	0.350	0.291	0.32	-
164	0.356	0.301	0.33	-
166	0.357	0.303	0.34	-

Table 10: The quadrupole deformations β_2 for Dy isotopes. See Table 7 for details.

A	NL-SH	FRDM	ETF-SI	expt.
142	0.250	0.219	0.32	-
144	-0.161	-0.164	-0.21	-
146	0.067	0.000	0.06	-
148	0.000	0.000	0.02	-
150	0.077	0.000	0.12	-
152	0.148	0.153	0.21	0.086
154	0.187	0.207	0.24	0.237
156	0.236	0.235	0.26	0.293
158	0.284	0.262	0.29	0.326
160	0.299	0.272	0.30	0.337
162	0.320	0.281	0.32	0.341
164	0.335	0.292	0.33	0.348
166	0.349	0.293	0.33	-
168	0.345	0.304	0.33	-

Table 11: The quadrupole deformations β_2 for Er isotopes. See Table 7 for details.

A	NL-SH	FRDM	ETF-SI	expt.
142	0.282	0.277	0.35	-
144	0.246	0.220	0.33	-
146	-0.166	-0.173	-0.21	-
148	-0.131	-0.156	-0.20	-
150	0.000	-0.008	0.02	-
152	0.080	-0.018	0.12	-
154	0.139	0.143	0.18	-
156	0.175	0.189	0.24	0.189
158	0.229	0.216	0.26	0.254
160	0.266	0.253	0.28	0.303
162	0.289	0.272	0.31	0.322
164	0.305	0.273	0.33	0.333
166	0.319	0.283	0.31	0.342
168	0.333	0.294	0.33	0.338
170	0.339	0.296	0.33	0.336

Table 12: The quadrupole deformations β_2 for Yb isotopes. See Table 7 for details.

A	NL-SH	FRDM	ETF-SI	expt.
154	0.091	-0.008	0.12	-
156	0.134	0.125	0.16	-
158	0.165	0.161	0.21	0.193
160	0.206	0.208	0.25	0.222
162	0.245	0.225	0.28	0.262
164	0.279	0.264	0.29	0.289
166	0.302	0.274	0.31	0.312
168	0.311	0.284	0.31	0.327
170	0.310	0.295	0.33	0.323
172	0.308	0.296	0.32	0.330
174	0.305	0.287	0.33	0.325
176	0.303	0.278	0.31	0.309
178	0.296	0.279	0.31	-
180	0.289	0.279	0.30	-
182	0.278	0.272	0.29	-
184	0.232	0.233	0.28	-

Table 13: The hexadecapole deformations β_4 for Nd isotopes obtained in the RMF theory using the force NL-SH. The FRDM and ETF-SI predictions for β_4 are also shown.

A	β_4		
	NL-SH	FRDM	ETF-SI
130	-0.010	0.002	0.020
132	-0.024	-0.002	0.020
134	-0.033	-0.023	0.030
136	-0.033	-0.030	-0.010
138	-0.009	-0.031	0.000
140	0.000	0.000	0.000
142	0.000	0.000	-0.010
144	0.000	0.000	0.010
146	0.009	0.068	0.040
148	0.033	0.083	0.050
150	0.093	0.107	0.070
152	0.114	0.128	0.070
154	0.104	0.114	0.070
156	0.085	0.098	0.070
158	0.067	0.082	0.050
160	0.041	0.069	0.030
162	0.020	0.048	0.030

Table 14: The hexadecapole deformations β_4 for Sm isotopes. See Table 13 for details.

A	β_4		
	NL-SH	FRDM	ETF-SI
134	-0.019	-0.006	0.020
136	-0.027	-0.021	0.020
138	-0.029	-0.037	0.000
140	-0.007	-0.030	0.000
142	0.000	0.000	0 000
144	0.000	0.000	-0.010
146	0.000	0.000	0.010
148	0.040	0.059	0.030
150	0.053	0.067	0.050
152	0.081	0.090	0.050
154	0.098	0.113	0.060
156	0.090	0.098	0.060
158	0.070	0.082	0.050
160	0.055	0.069	0.050
162	0.036	0.047	0.030
164	0.019	0.031	0.030

Table 15: The hexadecapole deformations β_4 for Gd isotopes. See Table 13 for details.

A	β_4		
	NL-SH	FRDM	ETF-SI
138	-0.028	-0.036	0.000
140	-0.012	-0.043	-0.010
142	-0.005	-0.029	-0.030
144	0.011	0.000	0.000
146	0.000	0.000	0.000
148	0.015	0.000	0.010
150	0.051	0.050	0.030
152	0.058	0.050	0.050
154	0.076	0.073	0.050
156	0.083	0.088	0.050
158	0.074	0.079	0.050
160	0.060	0.065	0.050
162	0.043	0.043	0.030
164	0.022	0.029	0.020
166	0.004	0.005	0.020

Table 16: The hexadecapole deformations β_4 for Dy isotopes. See Table 13 for details.

A	β_4		
	NL-SH	FRDM	ETF-SI
142	-0.044	-0.049	-0.030
144	-0.017	-0.028	-0.030
146	-0.001	0.000	0.000
148	0.000	0.000	0.000
150	0.024	0.000	0.010
152	0.046	0.041	0.030
154	0.051	0.041	0.030
156	0.065	0.046	0.040
158	0.080	0.060	0.050
160	0.070	0.053	0.040
162	0.056	0.040	0.030
164	0.035	0.025	0.020
166	0.016	0.010	0.020
168	-0.001	-0.012	-0.010

Table 17: The hexadecapole deformations β_4 for Er isotopes. See Table 13 for details.

A	β_4		
	NL-SH	FRDM	ETF-SI
142	-0.064	-0.073	-0.040
144	-0.056	-0.066	-0.030
146	-0.022	-0.035	-0.030
148	-0.025	-0.037	-0.040
150	0.000	0.000	0.000
152	0.015	0.000	0.010
154	0.030	0.040	0.020
156	0.033	0.030	0.010
158	0.053	0.034	0.020
160	0.068	0.040	0.010
162	0.062	0.037	0.020
164	0.050	0.020	0.020
166	0.033	0.006	0.000
168	0.006	-0.007	-0.001
170	-0.019	-0.023	-0.010

Table 18: The hexadecapole deformations β_4 for Yb isotopes. See Table 13 for details.

A	β_4		
	NL-SH	FRDM	ETF-SI
154	0.008	0.000	0.010
156	0.016	0.030	0.010
158	0.014	0.034	0.000
160	0.015	0.016	0.010
162	0.041	0.019	0.000
164	0.049	0.010	0.010
166	0.045	0.003	0.000
168	0.030	-0.010	0.000
170	0.005	-0.025	-0.010
172	-0.017	-0.040	-0.030
174	-0.036	-0.059	-0.030
176	-0.059	-0.071	-0.050
178	-0.074	-0.087	-0.050
180	-0.090	-0.098	-0.080
182	-0.102	-0.117	-0.070
184	-0.084	-0.128	-0.080

Table 19: The charge hexadecapole moment h_c in b^2 for Nd isotopes obtained in the RMF theory. The experimental data available (for details see text) is also given for comparison.

A	${h_c}^{RMF}$	${h_c}^{expt}$
130	0.152	-
132	0.070	-
134	0.021	-
136	-0.013	-
138	0.003	-
140	0.000	-
142	0.000	-
144	0.000	-
146	0.023	-
148	0.099	$0.36^{+0.10}_{-0.12}$
150	0.376	$0.30^{+0.06}_{-0.07}$, 0.25(12)
152	0.510	-
154	0.508	-
156	0.465	-
158	0.439	-
160	0.365	-
162	0.310	-

Table 20: The charge hexadecapole moments h_c for Sm isotopes. See Table 19 for details.

A	$h_c{}^{RMF}$	$h_c{}^{expt}$
134	0.114	-
136	0.064	-
138	0.021	-
140	0.027	-
142	0.000	-
144	0.000	-
146	0.000	-
148	0.124	-
150	0.193	-
152	0.359	0.46(2), 0.40(9), 0.37(8)
154	0.468	0.48(8), 0.63(5), $0.50^{+0.09}_{-0.08}$
156	0.469	-
158	0.432	-
160	0.405	-
162	0.366	-
164	0.319	-

Table 21: The charge hexadecapole moments h_c for Gd isotopes. See Table 19 for details.

A	$h_c{}^{RMF}$	$h_c{}^{expt}$
138	0.319	-
140	0.142	-
142	0.050	-
144	0.055	-
146	0.000	-
148	0.044	-
150	0.185	-
152	0.239	-
154	0.366	$0.38(16)$, $0.53(7)$, $0.64^{+0.06}_{-0.49}$
156	0.445	$0.42(8)$, $0.50(4)$, $0.41^{+0.12}_{-0.18}$
158	0.440	$0.39(9)$, $0.35(13)$, $0.34^{+0.20}_{-0.22}$
160	0.419	$0.36(10)$, $0.35^{+0.09}_{-0.07}$
162	0.385	-
164	0.339	-
166	0.292	-

Table 22: The charge hexadecapole moments h_c for Dy isotopes. See Table 19 for details.

A	$h_c{}^{RMF}$	$h_c{}^{expt}$
142	-0.001	-
144	0.028	-
146	0.012	-
148	0.000	-
150	0.082	-
152	0.185	-
154	0.228	-
156	0.313	$0.21^{+0.16}_{-0.20}$
158	0.419	$0.16\ ^{+0.10}_{-0.15}$
160	0.407	-
162	0.392	0.27(10)
164	0.338	0.28(10), 0.25(16), $0.23^{+0.10}_{-0.12}$
166	0.313	-
168	0.252	-

Table 23: The charge hexadecapole moments h_c for Er isotopes. See Table 19 for details.

A	$h_c{}^{RMF}$	$h_c{}^{expt}$
142	-0.050	-
144	-0.053	-
146	0.018	-
148	-0.011	-
150	0.000	-
152	0.061	-
154	0.134	-
156	0.164	-
158	0.271	-
160	0.366	-
162	0.377	$0.16^{+0.14}_{-0.26}$
164	0.361	$0.12^{+0.12}_{-0.13}$
166	0.323	$0.30(2)$, $0.32(16)$, $0.22^{+0.11}_{-0.16}$
168	0.248	$0.18(2)$, $0.20^{+0.12}_{-0.18}$
170	0.165	$0.31(2)$, $0.24^{+0.14}_{-0.18}$

Table 24: The charge hexadecapole moments h_c for Yb isotopes. See Table 19 for details.

A	$h_c{}^{RMF}$	$h_c{}^{expt}$
154	0.042	-
156	0.086	-
158	0.096	-
160	0.122	-
162	0.244	-
164	0.324	-
166	0.349	-
168	0.307	$0.19^{+0.14}_{-0.19}$, $+0.10^{+0.10}_{-0.09}$
170	0.214	-
172	0.130	$0.22^{+0.12}_{-0.18}$
174	0.058	$0.23(17)$, $0.22^{+0.14}_{-0.18}$
176	-0.029	-
178	-0.086	-
180	-0.149	-
182	-0.196	-
184	-0.162	-